

ON THE OVERLAPPINGS IN THE UNFOLDINGS OF THE DODECAHEDRON

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Abstract

No polyhedra have been found on which does not have a net, though most of its possible unfoldings overlap. In the particular case of the dodecahedron, the total number of unfoldings is 43380. We state by means of purely theoretical considerations that every unfolding is a net.

1. Introduction

An unfolding of a polyhedron consist of cutting it open along its edges in such a way that you can open it out in a single piece and flatten it out in the plane.

There are some interesting applications related to unfoldable connected structures. See, for instance, <http://www.patentstorm.us/patents/6920733-description.html>.

In 1975, Shephard conjectured that every convex polyhedron has an

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unfolding into a non-self-intersecting polygon [3]. There are non convex polyhedra for which every unfolding overlaps. On the other hand, this has lead to another question, there has not been found a polyhedron, which does not have a net, although most of its unfoldings overlap [4].

In the particular case of the dodecahedron, one can ask if it has an overlapping unfolding. The total number of unfoldings is 43380 [1, 2]. Yet this is a quite big number to look at by direct inspection. We will give an answer to this question by theoretical considerations.

2. Preliminaries

We give some definitions, which will help to develop our theoretical framework.

The regular dodecahedron P is the Platonic solid composed of 20 polyhedron vertices, 30 polyhedron edges ($E(P)$), and 12 pentagonal faces ($F(P)$).

Definition 2.1. A spanning tree of a graph $G = (V, E)$ is a connected subgraph of G with vertex set V , which does not contain a cycle.

Definition 2.2. An unfolding of the dodecahedron P with spanning tree A of the dual graphic of P , $D(P)$, is the function

$$\phi : P \rightarrow 2^{\mathbb{R}^2},$$

such that for each $f \in F(P)$, $\phi|_f : f \rightarrow \mathbb{R}^2$ is an isometric linear function and for every edge $e \in E(P)$ with $(f_1, f_2) = e^* \in A$, the following two conditions are satisfied:

- (i) $\phi|_{f_1}(p) = \phi|_{f_2}(p) \forall p \in e$.
- (ii) $\phi|_{f_1}(f_1) \cap \phi|_{f_2}(f_2) = \phi|_{f_2}(f_2 \cap f_1) = \phi|_{f_1}(f_2 \cap f_1)$.

The map ϕ is defined by

$$\phi(p) = \bigcup_{f \in F(P): p \in f} \phi|_f(p).$$

It is easy to see that an unfolding can be associated with the structure of a tree, precisely the spanning tree A defining the unfolding.

3. Theoretical Framework

Definition 3.1. A single chain net is a set of pentagons belonging to an unfolding and such that any of its elements shares at least one edge and at most two edges. The number of pentagons is called *the length of the single chain net*.

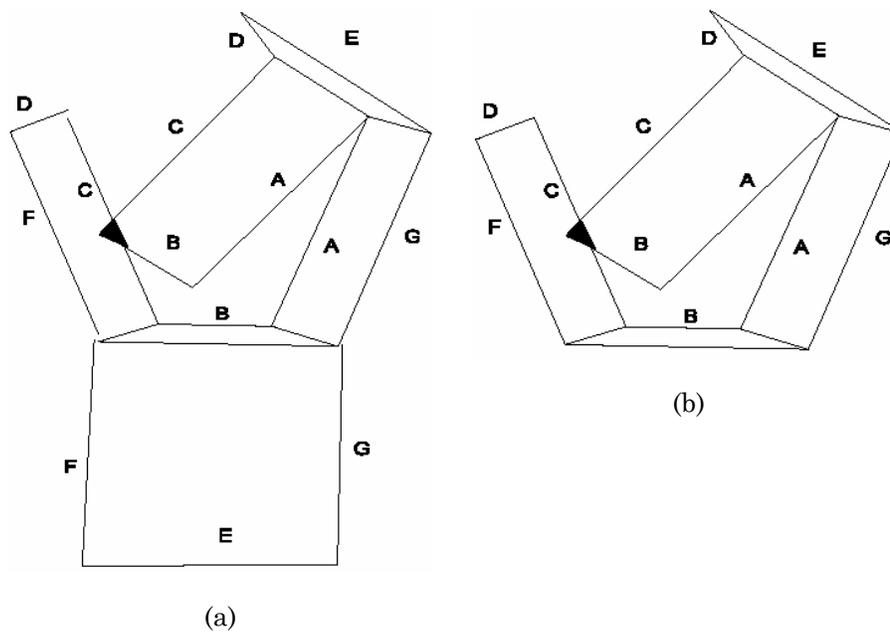


Figure 1. Figure (a) shows an unfolding of an hexahedron. Figure (b) contains the faces determining the overlapping.

It is a rather direct conclusion that every single chain net of length less than four does not overlap. Moreover, one can check that the only single chain nets, up to Euclidean transformations, of length four are given in Figure 2.

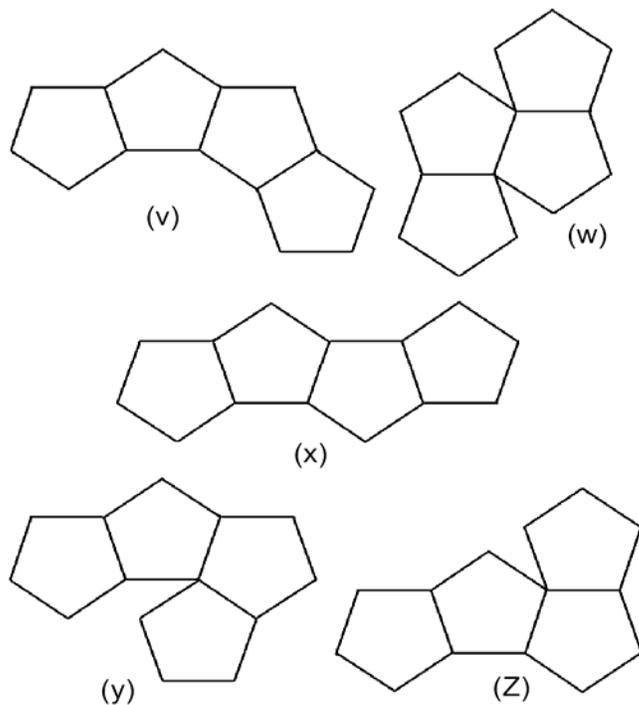


Figure 2. Basic configurations entering into every unfolding.

Our aim is to analyze all the single chain nets of lengths five to twelve. In doing so, we will describe a procedure to obtain a suitable set that contains, up to Euclidean transformations, all single chain nets.

Conceptually, to obtain every single chain net of lengths five to twelve, we must join to each of the five initial single chain nets of length four of the previous figure, another single chain net of length up to eight. Instead of this, we find it convenient to add on each side of the initial single chain net, another single chain net of adequate length.

An estimation of the size of the problem is given by the number of single chain nets of length twelve, these are, up to Euclidean transformations, 632, plus the other ones of lesser length. This is, however, a substantial reduction if we compare it with the number of unfoldings (43380).

We will use the dual graph of the dodecahedron to characterize all the single chain nets. Figure 3 shows this dual graph.

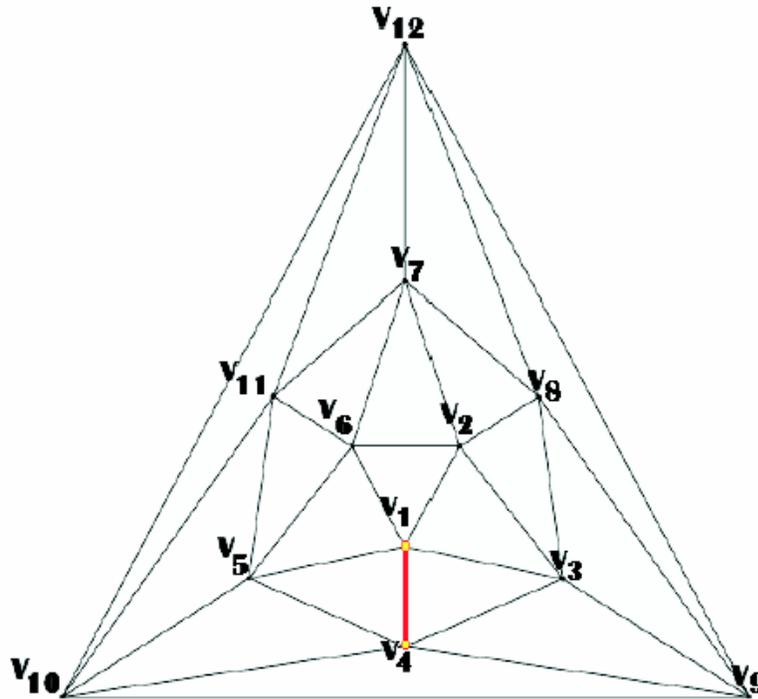


Figure 3. Dual graph of the dodecahedron.

Definition 3.2. A path in a graph G is a sequence of edges

$$\overline{v_0v_1}, \overline{v_1v_2}, \dots, \overline{v_{k-1}v_k},$$

such that $v = v_0$, $v_k = w$, and v_0, \dots, v_k are all distinct. The length of the path is the number of edges in it.

Figure 4 shows a search tree enumerating all paths starting from vertex v_1 , including edge $\overline{v_1v_4}$ of length ≤ 4 in the dual graph of the dodecahedron. The number of leaves at the search tree is 46.

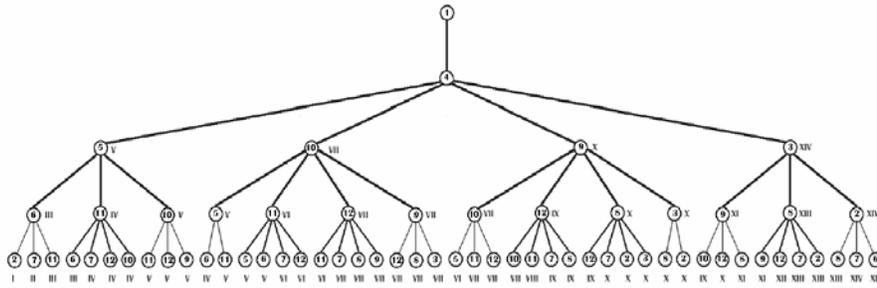


Figure 4. Search tree of paths of length four.

Each path corresponds to a specific configuration of pentagons immersed into an unfolding. For example, Figure 5 shows an arrangement of pentagons, which is associated to the path $\overline{v_1v_4}, \overline{v_4v_{10}}, \overline{v_{10}v_5}, \overline{v_5v_6}$.

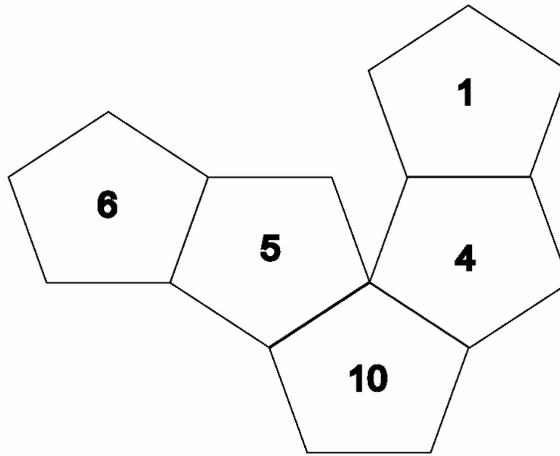


Figure 5. Configuration of pentagons associated to path $\overline{v_1v_4}, \overline{v_4v_{10}}, \overline{v_{10}v_5}, \overline{v_5v_6}$.

It is convenient first to give an order to the search tree in Figure 4. Figure 6 shows a collection of semi-planes determined by each line starting at four strategic points at the first two pentagons of the extreme

of every single chain net. These 14 lines define 14 semi-planes Π_i^+ , $1 \leq i \leq 14$. Π_i^+ is the region swept by counterclockwise rotation, with rotation center at a point of the line, by an angle less than or equal to π .

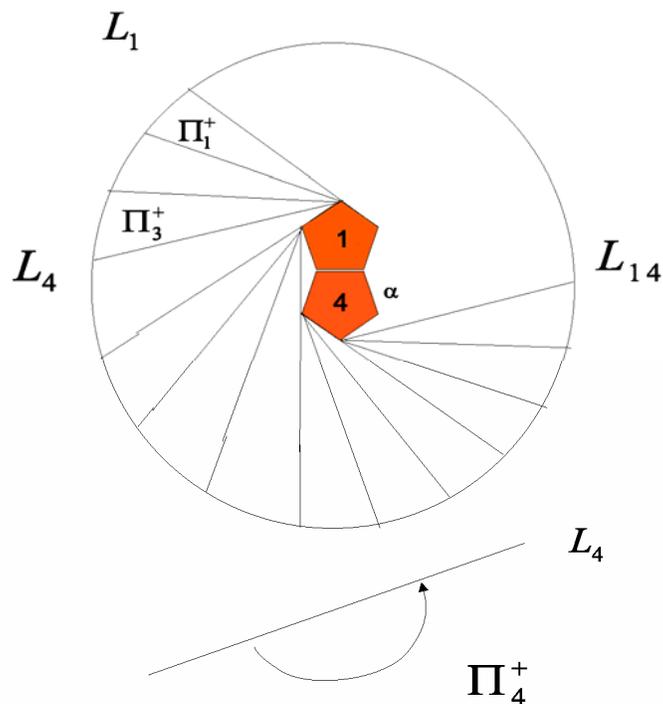


Figure 6. The semi-planes Π_i^+ , $1 \leq i \leq 14$.

Once one has defined these semi-planes, we order the single chain nets of the search tree according to a bucket sort.

Definition 3.3. We say that a single chain net τ corresponding to a branch of the search tree is in the bucket indexed i , if it satisfies the following two conditions:

- (i) τ is completely contained in the semi-plane Π_i^+ .
- (ii) τ is not contained in the semi-plane Π_{i+1}^+ .

One deduces that every single chain net corresponding to a branch of the search tree is assigned to a unique bucket. Figure 7 shows the respective order according to the semi-planes Π_i^+ , $1 \leq i \leq 14$, assigned to every single chain net represented in the search tree.

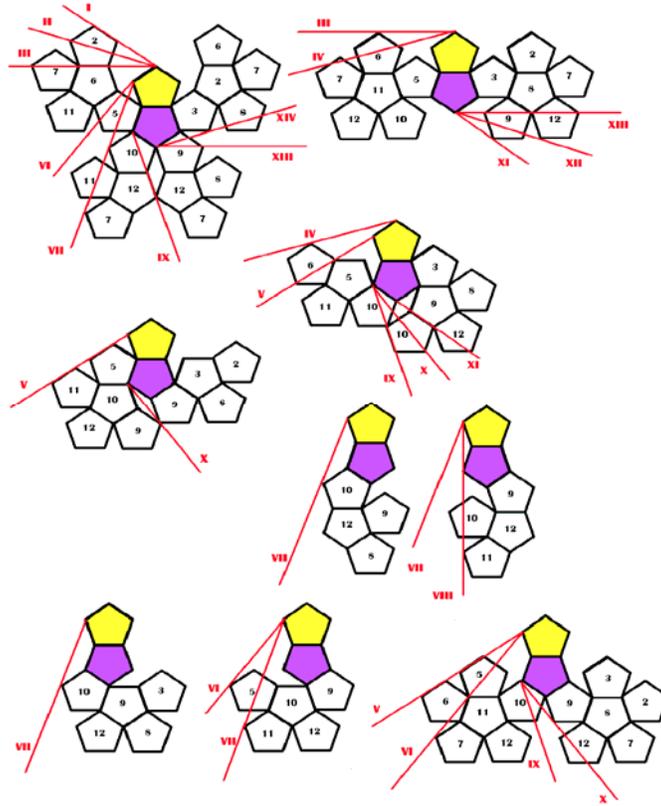


Figure 7. Classification of single chain nets of length four.

Now, we analyze all single chain nets of length less or equal to six and starting with pentagons 1 and 4 (see Figure 6). Note that this leads to consider a search tree of about 184 leaves, because we can attach a single chain net of length four at four sides of pentagon 4. Since our objective is to discard an overlapping, one does not need to determine exactly each of those single chain nets. Instead of this, what we do is to fix a region, where these single chain nets are contained.

In doing so, we proceed iteratively. Suppose, we first study those single chain nets by using the side marked (α) of pentagon four of Figure 6. The pentagon attached at side (α) must be marked three, and we can make use of our previous analysis to classify all these single chain nets. One needs only to identify these pentagons 4 and 3 with the pentagons 1 and 4 of Figure 6, respectively. See Figure 9.

This procedure must be done at the other three sides of pentagon four in Figure 6, in order to analyze all single chain nets of length less or equal to six and starting with pentagons 1 and 4. The details are discussed below.

Note that for reasons of symmetry, it is always possible to identify any two adjacent pentagons in an unfolding with two adjacent faces of the dodecahedron according to some (fixed) numeration of its faces. We have chosen, in this paper, the one given in Figure 8. Observe also that, once we have identified two pentagons in an unfolding (in particular also in a single chain net) with two faces in that figure, then all the other faces to be identified at the unfolding are determined.

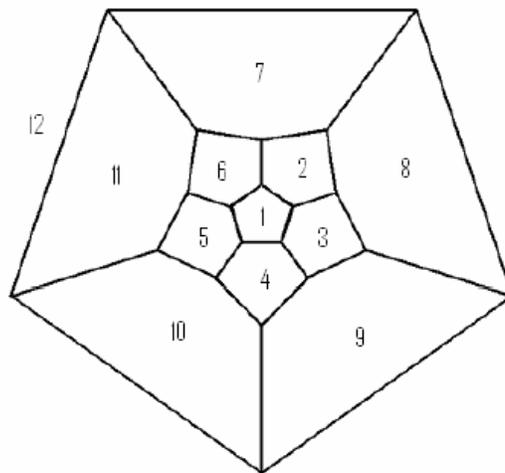


Figure 8. Labelling of the faces of the dodecahedron.

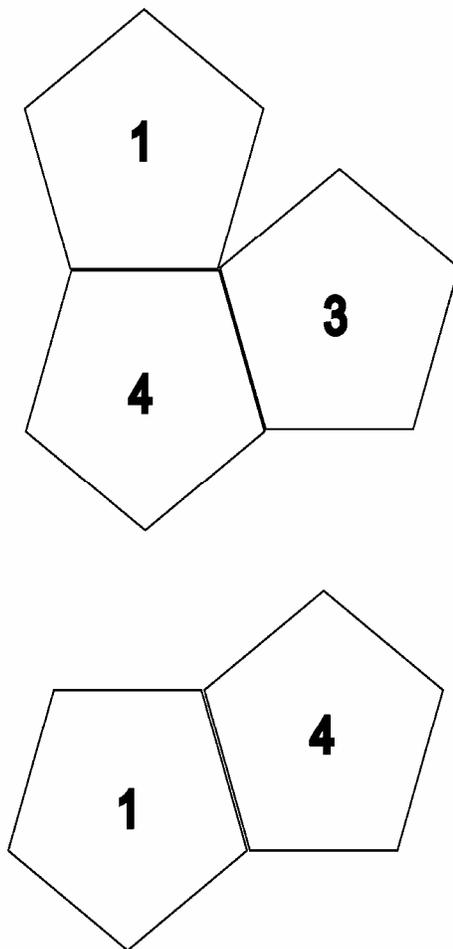


Figure 9. Analysis of single chain nets of length less or equal to six.

Taking into account these considerations, we will use the expression *modulo a labelling of the pentagons* to indicate that a statement is valid after identifying all the pentagons in a single chain net with faces of the dodecahedron.

Lemma 3.1. *Suppose that we have four pentagons as shown in Figure 10, where we imagine Figure 6 rotated and translated so that pentagons one and four coincide with pentagons b and c of Figure 10. To simplify*

notation, we denote by the same symbols L_j and Π_j^+ , $j = 1, \dots, 14$, the corresponding lines and sectors obtained by application of this Euclidean transformation in Figure 6. Then, for every single chain net N containing pentagons a, b, c and being in some bucket Π_ℓ^+ , there exists at least a single chain net N_+ containing pentagons a, b, d and being in some bucket Π_j^+ with $j \leq \ell$. Furthermore, modulo a labelling of the pentagons, N_+ can be chosen so that all the pentagons except perhaps pentagon d forming it, are also in N .

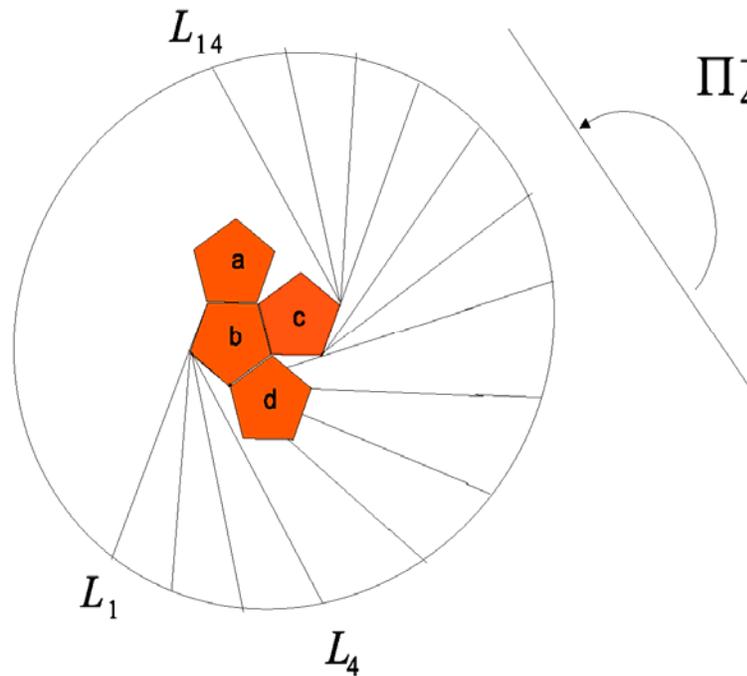


Figure 10. Graphic for Lemma 3.1.

Proof. We refer to Figure 10. Note that the statement of the lemma is obvious for the single chain nets having pentagons a, b, c and being in buckets Π_j^+ , $j \geq 4$. For each single chain net N of such nets, we take the

net N_+ as the net consisting of the pentagons a, b, and d, which obeys the assertions of the lemma. On the other hand, there are only five single chain nets being in the buckets Π_j^+ , $j = 1, 2, 3$. Modulo a labelling of the pentagons, pentagons a, b, c, d correspond to pentagons 3, 1, 4, 5, respectively (see Figure 8). From Figure 4, we obtain that the five nets are $3-1-4-5-6-2$, $3-1-4-5-6-7$, $3-1-4-5-6-11$, $3-1-4-5-6$, and $3-1-4-5-11-6$. For these nets, N_+ can be taken as $3-1-5-6$. This proves the lemma. \square

Similarly, we can prove the following two lemmas:

Lemma 3.2. *Suppose that we have four pentagons as shown in Figure 11, where we imagine Figure 6 rotated and translated so that pentagons one and four coincide with pentagons b and d of Figure 11. Then, for every single chain net containing pentagons a, b, d and being in some bucket Π_ℓ^+ , there exists at least a single chain net containing pentagons a, b, e and being in some bucket Π_j^+ with $j \leq \ell$. Furthermore, modulo a labelling of the pentagons, N_+ can be chosen so that all the pentagons with the possible exception of pentagon e forming it, are also in N .*

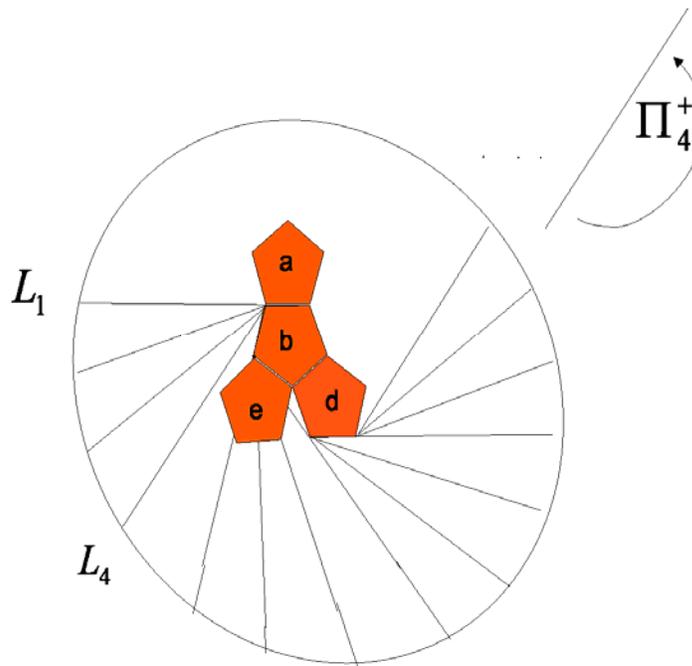


Figure 11. Graphic for Lemma 3.2.

Lemma 3.3. *Suppose that we have four pentagons as shown in Figure 12, where we imagine Figure 6 rotated and translated so that pentagons one and four coincide with pentagons b and e of Figure 12. Then, for every single chain net containing pentagons a, b, e and being in some bucket Π_ℓ^+ , there exists at least a single chain net containing pentagons a, b, f and being in some bucket Π_j^+ with $j \leq \ell$. Furthermore, modulo a labelling of the pentagons, N_+ can be chosen so that all the pentagons, but perhaps pentagon f forming it, are also in N .*

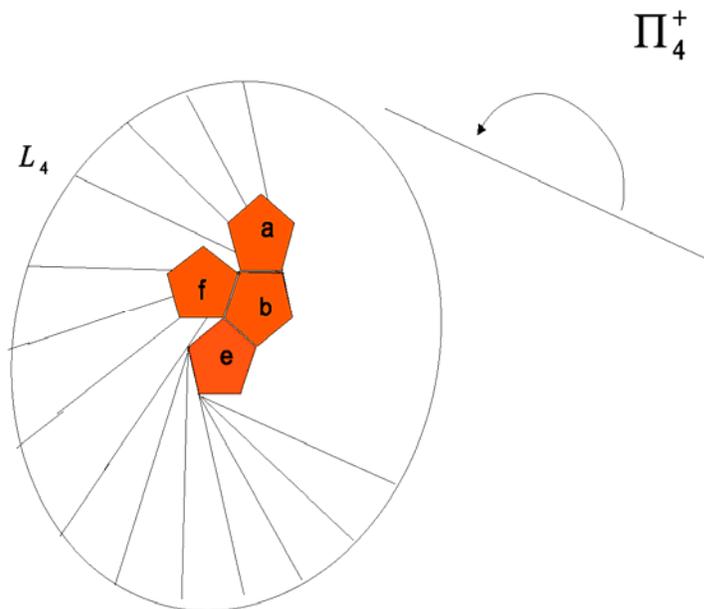


Figure 12. Graphic for Lemma 3.3.

4. Main Theorem

Lemma 4.1. *Every unfolding of the dodecahedron with an overlapping has a self intersecting single chain net.*

Proof. Suppose that \mathcal{U} is an unfolding of the dodecahedron having an overlapping. Let v_1 and v_k be a pair of facets of \mathcal{U} with non void intersection and not being adjacent in \mathcal{U} . Since every unfolding is a connected set, there must exist a subset $\{v_1, \dots, v_k\}$ of \mathcal{U} forming a single chain net. This proves the lemma. \square

Theorem 4.1. *If every single chain net of the dodecahedron does not intersect itself, then every unfolding of it does not overlap.*

Proof. Since every unfolding that overlaps contains a self intersecting single chain net, from the previous lemma, the result follows at once. \square

Now we consider a few cases that appear in our analysis, together with a symmetry argument simplifying the proof of the main theorem.

Theorem 4.2. *No unfolding of the dodecahedron overlaps.*

Proof. Figure 2 shows the five basic configurations that enter into every unfolding of length four. Furthermore, to obtain every single chain net of lengths five to twelve, we must just join to each of these five initial single chain nets of length four, another single chain net of length up to eight. As previously mentioned, we find it convenient to attach at each side of the initial single chain net, another single chain net of adequate length. We have five cases, corresponding to each one of the five configurations shown in Figure 2. Since each case is treated basically in the same way, we give the details only for one of them.

Figure 13 shows the basic configuration 2 (w) (with the pentagons d and f attached). We want to fix a region U containing all single chain nets of length up to four attached at pentagon δ to this basic configuration, and forming a new single chain net of length up to eight.

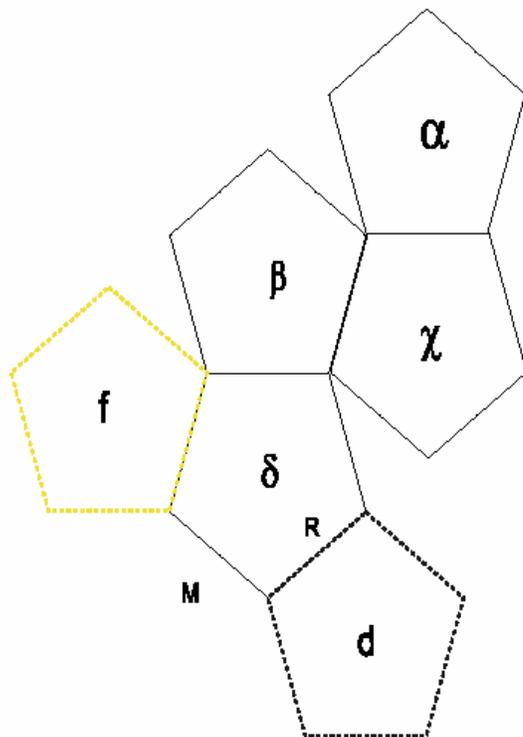


Figure 13. Basic configuration of pentagons marked (w) in Figure 2 with pentagons d and f added.

First, we apply Lemma 3.3, by identifying pentagons b and f in Figure 12 with pentagons δ and f in Figure 13. Then, every single chain net of length less or equal to four, attached to pentagon δ at side M, lies in a bucket containing also a single chain net starting with pentagon f.

Now, by using Lemma 3.2, one concludes that the single chain nets starting with pentagon d are delimited by the buckets, where the ones attached to side M lie. Therefore, by transitivity, we can assure that the region U is delimited by the single chain nets starting with pentagon f. We proceed to state precisely, which single chain nets are the ones in question.

If we identify pentagons 1 and 4 with pentagons δ and f in Figure 13, respectively, then we obtain Figure 14(A) (pentagon d suppressed). When we intend now to form a single chain net by attaching to configuration 14(A), another single chain net at pentagon 4, we must avoid to use pentagons labelled 5, 6, and 11. From Figure 4, we obtain the branches that use nodes 5, 6, and 11, which have to be rejected.

It follows that the buckets Π_j^+ covering region U are those with $j \geq 7$. See Figure 14(A). (Figure 6 has been rotated and translated so that pentagons 1, 4 coincide with the ones in Figure 14(A).)

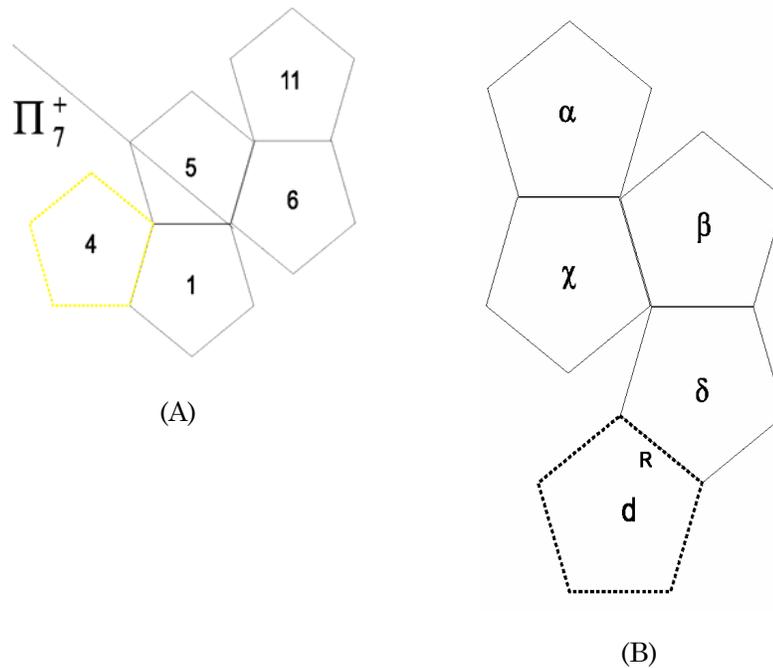


Figure 14. Graph (A) shows the basic configuration 2 (w) after identification of its pentagons. Graph (B) shows a reflection of this configuration.

This argument proves that the region U is bounded in the “clockwise” direction.

We wish now to bound the region U in the “counterclockwise” direction. This can be done by applying the same Lemmas 3.1-3.3 together with a symmetry argument.

A reflection of Figure 13 is given by Figure 14(B). In the reflection, the corresponding pentagon f has been dropped. From Lemmas 3.1 and 3.2, we obtain as before that all single chain nets of length up to four attached at pentagon δ in Figure 14(B) are bounded by the ones starting with pentagon d .

Now, we proceed to characterize these single chain nets. If we identify pentagons 1 and 4 with pentagons δ and d in Figure 14(B), respectively, then we obtain Figure 15(A) (Figure 6 has been rotated and translated so that pentagons 1, 4 coincide with the ones in Figure 15(A)).

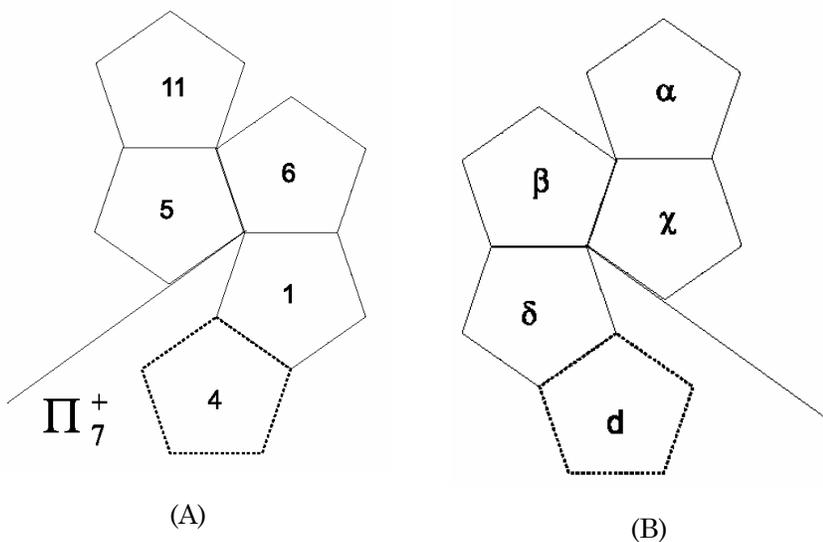


Figure 15. Graph (A) shows the region containing all single chain nets of length up to four attached to pentagon δ , for the configuration 14(B) after identification of its pentagons. Graph (B) shows a reflection of this configuration.

When we intend now to form a single chain net by attaching to configuration 15(A), another single chain net at pentagon 4, we must

discard pentagons labelled 5, 6, and 11. Looking at Figure 4, we see that the single chain nets with pentagon d lie in the buckets Π_j^+ with $j \geq 7$. See Figure 15(A).

A reflection of this figure gives a bound for the region U in the “counterclockwise” direction, as Figure 15(B) shows.

Therefore, the region U is contained between the lines shown in Figure 16(A).

To finish the analysis for the basic configuration 2 (w), we must fix also the region U' containing all single chain nets of length up to four, attached at pentagon α . This one sees at once noting that a reflection along a horizontal line followed by a reflection along a vertical line leaves the configuration 2 (w) invariant. Then, the region U' is as shown in Figure 16(B). This proves that every single chain net of length up to twelve and containing the basic configuration 2 (w) does not overlap.

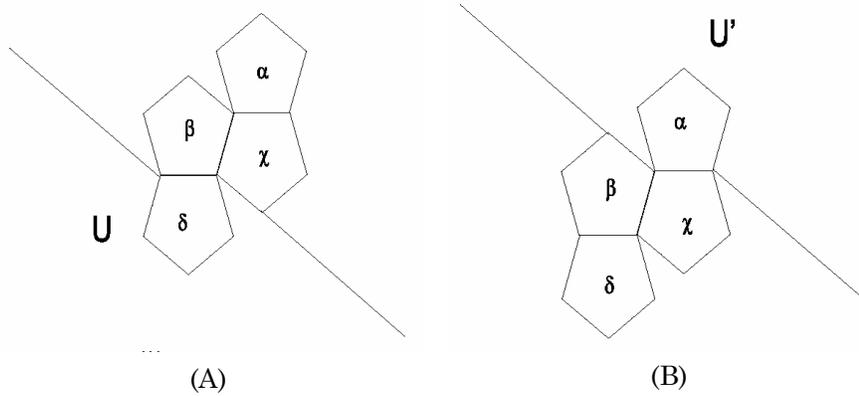


Figure 16. Graphic (A) shows region U delimited by the two lines. Graphic (B) shows the region U' obtained by application of two reflections of region U .

The proof is complete after doing the same analysis for the other four basic configurations in Figure 2. This analysis is exactly the same as the one discussed previously, being however a rather direct calculation. \square

References

- [1] F. Buekenhout and M. Parker, The number of nets of the regular convex polytopes in dimension ≤ 4 , *Discrete Math.* 186(1-3) (1998), 69-94.
- [2] Ch. Hippenmeyer, Die Anzahl der inkongruenten ebenen Netze eines regulren Ikosaeders, *Elem. Math.* 34 (1979), 61-63.
- [3] G. C. Shephard, Convex polytopes with convex nets, *Math. Proc. Cambridge Philos. Soc.* 78(3) (1975), 389-403.
- [4] W. Schlickenrieder, Nets of Polyhedra, PhD. Thesis, Technische Universität, Berlin, 1997.

